where the levels m refer to an element in ionization stage z. Clearly, this cutoff would have added to the consistency of the McGee-Heller calculations if it had been used. It appears therefore that the Debye-Huckel theory of point charges provides a consistent model for medium-density, medium-temperature plasmas. What model therefore is appropriate for denser plasmas, where the Debye validity criteria are invalid and into which regime the McGee and Heller calculations likely encroach? If it is desired to extend the pressure range beyond the range of the Debye criteria, then the theory must be generalized to avoid not only the divergence of the collision integrals at long range (Debye length) but also the divergence at short range. This latter divergence can be removed by assuming that the ions have a finite, nonzero diameter; this point is discussed by Duclos and Cambel. Now it will be recalled that Debye-Huckel theory avoided the statistical aspects of the ionized medium by assuming that the electrostatic potential obeyed the Poisson-Boltzmann equation. When linearized, the resulting equation could be solved exactly and yielded the thermodynamic functions used by McGee and Heller. However, when finite ionic diameters are introduced into the linearized equation, inconsistencies arise which unfortunately do not disappear on removal of the linearization. To avoid these difficulties, Meyer[11] made a fresh statistical approach (cluster theory using ring-integrals), and this theory has been extended by several workers; their results recently have been discussed by Brush, DeWitt, and Trulio.[12] These analyses show that the pressure decrease predicted by Debye-Huckel theory of point charges is overestimated if finite ionic diameters are considered. The problem of pressure ionization becomes relevant here, viz., at a fixed temperature the degree of ionization of a gas decreases with pressure as decreed by the Saha equation but then passes through a minimum and then increases with pressure. Pressure ionization that is attributed to the squeezing of the electronic levels of an atom into the continuum due to increasing atom density in the immediate vicinity has been treated simply by Bethe and more exactly by Timan[13] using the Debye-Huckel theory of point charges. It also has been studied by Brush[14] for the case of finite ionic diameter; he found that the pressure p^*, where the degree of ionization became a minimum, was given by the approximate expression

$$p^* = \left[1 + \frac{4e(\pi p^*)^{1/2}}{\beta kT}\right]\left[\frac{k^4T^4}{4\pi e^6}\right]$$

where β^{-1} is the mean radius of the electronic charge distribution of an ion. The analysis giving this result is suitable for a gas with two stages of ionization but with a preponderance of one stage. If the value of 10^{-8} cm is taken for the mean ionic radius, then at a temperature of 10,000°K, the foregoing formula gives a pressure of 36 atm, whereas if the ionic radius is put equal to zero at the same temperature, then a pressure of 23.6 atm is found, i.e., the introduction of a finite ionic diameter increases the gas pressure above that predicted by the Debye theory of point charges. Bearing this in mind, it would appear that the higher-pressure, lower-temperature results of McGee and Heller for hydrogen and for lithium should be re-examined, since it is likely that their tabulated Debye-Huckel point charge excess functions will be in error. Strictly speaking, although there are well-developed theories for discussing the high-temperature, high-pressure gas, such as those reviewed by Brush et al.,[12] they all suffer from the one defect, viz., the gas imperfections that they predict are sensitive functions of the mean radius of the ion, and this quantity cannot be decided unambiguously at the present time.

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Integrated Laminar Heat Transfer in the Windward Plane of Yawed Blunt Cones

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THE ratio of local to stagnation heat-transfer rates on a blunt body in axisymmetric laminar flow is given by Ref. 1:

$$\frac{q}{q_0} = \frac{\rho_w\mu_w u_e r}{(2\xi)^{1/2}[2\rho_{w0}\mu_{w0}(du_e/ds)_0]^{1/2}} \frac{1 + 0.096(\beta)^{1/2}}{1.068} \left(\frac{H_0 - h_w}{H_0 - h_{w0}}\right) \tag{1}$$

where

$$\beta = 2(d \ln u_e/d \ln\xi) \tag{2}$$

and

$$\xi = \int_0^s \rho_w\mu_w \, u_e r^2 ds \tag{3}$$

with s being the distance along a streamline at the edge of the boundary layer.

According to Ref. 2, the heating-rate distribution over an axisymmetric body may be expressed by replacing the cylindrical radius r in Eqs. (1) and (3) by a curvilinear variable h_2. In the windward plane h_2 is expressed by the differential equation

$$\frac{1}{h_2}\frac{\partial h_2}{\partial s} = \frac{1}{r}\frac{\partial r}{\partial s} + \frac{1}{u_e}\frac{\partial w}{r\partial\phi} \tag{4}$$

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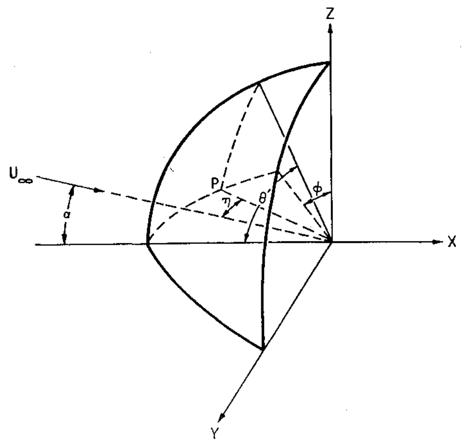

Fig. 1 Coordinate system

Using Eqs. (1) and (4) and assuming that Newtonian pressure distribution holds over the body, an integrated heating-rate distribution may be obtained over a spherically blunted cone.

For the geometrical configuration shown in Fig. 1, the pressure ratio and the velocity at any point P on the spherical nose are given by

$$u_e = U_\infty \sin\eta \qquad p/p_0 = \cos^2\eta \tag{5}$$

$$\cos\eta = \cos\alpha \cos\theta + \sin\alpha \sin\theta \cos\phi \tag{6}$$

The components of the freestream velocity vector in the x and z directions are

$$U_{\infty x} = U_\infty \cos\alpha \qquad U_{\infty z} = -U_\infty \sin\alpha \tag{7}$$

In a ϕ = const plane, the contributions of the velocity components to the meridional and circumferential surface velocities are

$$u_{\text{merid}} = U_\infty[\cos\alpha \sin\theta - \sin\alpha \cos\phi \cos\theta]$$
$$w = U_\infty \sin\alpha \sin\phi \tag{8}$$

Differentiating w with respect to ϕ,

$$\left.\frac{\partial w}{\partial \phi}\right|_{\phi=0} = U_\infty \sin\alpha$$

and substituting into Eq. (4), the differential equation for h_2 becomes

$$\frac{1}{h_2}\frac{dh_2}{ds} = \frac{1}{r}\frac{dr}{ds} + \frac{\sin\alpha}{R_b \sin\theta \sin(\theta - \alpha)} \tag{9}$$

The solution of Eq. (9) is

$$h_2 = R_b \sin(\theta - \alpha) \tag{10}$$

As might be expected, Eq. (10) shows that the effect of angle of attack on the heat transfer to a spherical surface is just to shift the origin of integration to the stagnation point. Therefore, the heat-transfer distribution on the spherical nose has a form similar to the one derived by Lees,[3] with θ replaced by $(\theta - \alpha)$.

For an isothermal wall, Eq. (3) may be integrated with the aid of Eqs. (5, 8, and 10) provided that the ratio of molecular weights at the wall is constant:

$$\xi = \int_0^s \rho_w \mu_w u_e h_2^2 ds = \frac{p_0 \mu_w}{Z_w R T_w}\int_0^s \frac{p_e}{p_0} u_e h_2^2 ds = \frac{R_b^3 p_0 \mu_w U_\infty}{Z_w R T_w} M(\theta - \alpha) \tag{11}$$

where

$$15M(\theta - \alpha) = [1 - \cos(\theta - \alpha)]^2[3\cos^3(\theta - \alpha) + 6\cos^2(\theta - \alpha) + 4\cos(\theta - \alpha) + 2] \tag{12}$$

Similarly, from Eqs. (2, 3, and 8),

$$\beta = 2\frac{1}{u_e}\frac{du_e}{ds}\xi\frac{ds}{d\xi} = 2\frac{M(\theta - \alpha)}{\sin^4(\theta - \alpha)\cos(\theta - \alpha)} \tag{13}$$

Substituting Eqs. (5, 8, 10, and 11) into Eq. (1) with r replaced by h_2,

$$\frac{q}{q_0} = \frac{\cos^2(\theta - \alpha)\sin^2(\theta - \alpha)}{2[M(\theta - \alpha)]^{1/2}}\;\frac{1 + 0.096\,(\beta)^{1/2}}{1.068} \tag{14}$$

Equation (14) expresses the ratio of local to stagnation-point heating rate over a spherical surface.

On the conical afterbody, Eq. (4) may be rewritten as

$$\frac{1}{h_2}\frac{dh_2}{ds'} = \frac{1}{s'}\left[1 + \frac{1}{\sin\theta_c u_e}\frac{\partial w}{\partial \phi}\right] \tag{15}$$

where s' is the distance from the hypothetical apex of the cone and is given by

$$s'/R_b = \cos\theta_c + [(s/R_b) - (\pi/2) + \theta_c + \alpha] \tag{16}$$

and the expression in the brackets in Eq. (15) is a constant on the cone if conical flow exists. Therefore, integrating Eq. (15) and introducing the boundary condition that at the junction of the cone and the sphere h_2 is given by Eq. (10), with $\theta = (\pi/2) - \theta_c$,

$$h_2 = R_b \cos(\theta_c + \alpha)[s'/R_b \cos\theta_c]^k \tag{17}$$

where

$$k = 1 + \frac{1}{u_e \sin\theta_c}\frac{\partial w}{\partial \phi} = \frac{\tan(\theta_c + \alpha)}{\tan\theta_c} \tag{17a}$$

Substituting Eq. (17) into Eq. (3),

$$\xi = \frac{R_b^3 p_0 \mu_w U_\infty}{Z_w R T_w}\left[M\left(\frac{\pi}{2} - \theta_c - \alpha\right) + A^3\frac{\sin^2(\theta_c + \alpha)\cos^3(\theta_c + \alpha)}{(\cos\theta_c)^{2k}}\int_{\cos\theta_c}^{s'/R_b}\left(\frac{s'}{R_b}\right)^{2k} d\left(\frac{s'}{R_b}\right)\right] =$$
$$\frac{R_b^3 p_0 \mu_w U_\infty A^3}{Z_w R T_w}\left\{\frac{M[(\pi/2) - \theta_c - \alpha]}{A^3} + \frac{\sin^2(\theta_c + \alpha)\cos^3(\theta_c + \alpha)}{(1 + 2k)(\cos\theta_c)^{2k}}\left[\left(\frac{s'}{R_b}\right)^{1+2k} - (\cos\theta_c)^{1+2k}\right]\right\} \tag{18}$$

The factor A appearing in Eq. (18) accounts for the fact that the pressure on the cone equals the asymptotic cone pressure rather than the Newtonian pressure. (A was found to be approximately 1.12.) On the cone, $\beta = 0$; therefore, substituting Eqs. (5, 8, and 18), together with the A factor, into Eq. (1) yields

$$\frac{q}{q_0} = \frac{A^{3/2}P(\theta_c + \alpha)\,[(s'/R_b)]^k}{1.068\,\{N(\theta_c, \alpha) + [(s'/R_b)]^{1+2k}\}^{1/2}} \tag{19}$$

with

$$N(\theta_c, \alpha) = \frac{(1 + 2k)(\cos\theta_c)^{2k}}{\sin^2(\theta_c + \alpha)\cos^3(\theta_c + \alpha)}\;\frac{M[(\pi/2) - \theta_c - \alpha]}{A^3} - (\cos\theta_c)^{1+2k} \tag{20}$$

$$P(\theta_c - \alpha) = [(1 + 2k)^{1/2}/2]\sin(\theta_c + \alpha)\cos^{1/2}(\theta_c + \alpha) \tag{21}$$

k is evaluated from Eq. (17a).

The heat-transfer rate given by Eq. (19) was evaluated for a 15°-half-angle blunt cone at various angles of attack and is compared with the experimental results presented in Refs. 4 and 5. The results are shown in Fig. 2.

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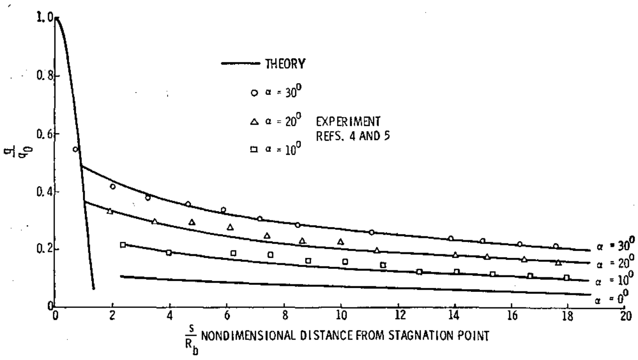

Fig. 2 Laminar heat-transfer rate ratio for a 15°-semi-angle blunt cone at three angles of attack

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Insulation Requirements for Long-Time Low-Heat Rate Environments

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Portions of lifting vehicles entering the earth's atmosphere at satellite and supersatellite speeds experience low heat rates for comparatively long durations. Heat rates of 10 to 50 Btu/ft²-sec for periods ranging up 2000 sec are not uncommon on afterbodies and leeward surfaces. In this environment, a thermal protection system of the type shown in Fig. 1 often is considered in preliminary design analyses. It consists of a low-density, low-conductivity inorganic foam applied to a metallic load-carrying substrate.

If the foam does not undergo thermal decomposition or experience oxidation reactions, the performance of such a system can be analyzed simply on the basis of a transient heat conduction analysis. This note presents an analysis that relates the weight per unit area of the insulation to the allowable temperature rise of the substrate in terms of the thermal properties of insulation and the heating environment.

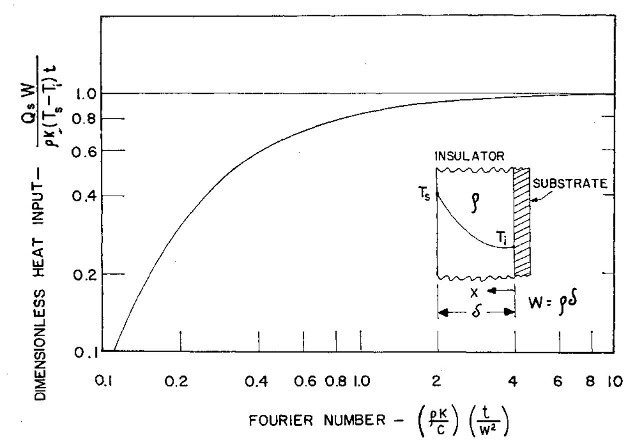

Fig. 1 Heat transmitted to rear surface of slab, the surfaces of which are maintained at constant temperatures

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The heating environment is characterized by a square heat pulse with a magnitude $\dot{q}$ and a duration t. Stagnation enthalpy is considered to be much greater than the wall enthalpy. The heated surface of the low-density, low-conductivity insulator rapidly reaches a temperature near the radiation equilibrium value $(\dot{q}/\epsilon\sigma)^{1/4}$. As heat diffuses through the insulation into the substrate, the temperature of the inner surface of the insulation gradually rises. Normally, however, the allowable temperature rise of the substrate is small in comparison to the heated surface temperature. An upper bound for the heat transmitted to the substrate can be obtained by solving the diffusion equation for the insulation, subject to the boundary conditions

$$T(0,t) = T_i$$
$$T(\delta,t) = T_s \simeq (\dot{q}/\epsilon\sigma)^{1/4} \qquad (1)$$
$$T(x,0) = T_i$$

Surprisingly, the solution to the transient heat conduction equation under the simple boundary conditions just given is not contained in standard heat conduction texts such as Carslaw and Jaeger. The solution was obtained by conventional methods to determine temperature-time histories in the insulation, subject to the assumption of constant thermal properties.

The quantity of primary interest is the integrated heat input to the substrate, Q_s. This is determined by differentiating the solution for $T(x,t)$ with respect to x, multiplying by the thermal conductivity, integrating with respect to time, and evaluating the result at the insulation-substrate interface. The result is

$$Q_s = \int_0^t k \left.\frac{\partial T}{\partial x}\right|_{x=0} d\tau = \frac{\rho k(T_s - T_i)t}{W} \times \left\{1 + \frac{2}{\pi^2(\rho k/c)(t/W^2)} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \times \left[1 - \exp\left(-(n\pi)^2 \frac{\rho k}{c}\frac{t}{W^2}\right)\right]\right\} \qquad (2)$$

where ρ, k, and c are the density, thermal conductivity, and specific heat of the insulation, respectively, and W is the section weight $\rho\delta$. The result expressed by Eq. (2) is shown in dimensionless form in Fig. 1. The Fourier number has been written in the form $(\rho k/c)(t/W^2)$ to emphasize the significant thermal property grouping when weight per unit area rather than thickness is used to characterize the insulation. In general, the pulse duration, surface temperature, and the allowable heat input to the substrate are considered known quantities, and the problem is to determine the insulation weight per unit area. This information can be obtained from the results shown on Fig. 1.

For Fourier numbers in excess of 3, the dimensionless heat input is close to unity, indicating that near-steady conduction prevails over the duration of the heat pulse. In this case, the required section weight is simply

$$W = \rho k(T_s - T_i)t/Q_s \qquad (3)$$

The foregoing result also indicates that, for long-time, low-heat rate conditions, the required section weight for a given insulator depends more strongly on the pulse duration and allowable heat input to the substrate than on the integrated cold wall heat input $\dot{q}t$, since

$$W = \frac{\rho k[(\dot{q}t/\epsilon\sigma)^{1/4} - T_i t^{1/4}]t^{3/4}}{Q_s} \simeq \frac{\rho k(\dot{q}t)^{1/4}t^{3/4}}{(\epsilon\sigma)^{1/4}Q_s} \qquad (4)$$